Signal Broadening in the Laser Doppler Velocimeter

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In a recent paper Denison, Stevenson, and Fox (*1*) discussed the sources of spectral broadening in the laser doppler velocimeter. In their discussion they indicated that the spread in wave vectors of the incident and detected fields *and* the finite length of time a scattering center stayed in the sample volume each contributed separately and independently to the observed spectral width of the scattered radiation. This statement, found also in other works (*2*), is incorrect and, moreover, can lead to erroneous interpretation of data. The errors involved can be very serious, for example, when using the observed spectral width to measure the percent turbulence. In this note we show that the two effects are one and the same.

First, consider the case where one is treating the phenomenon as a spread or uncertainty in wave vectors of the incident and scattered light. The calculation proceeds as follows. The distribution of light in the beams is modeled by a linear superposition of plane waves, each with a well-defined wave vector, $\mathbf{k}$. (The fact that one can make a three dimensional Fourier decomposition of an arbitrary radiation field into a linear combination of plane waves is a well-known result of electromagnetic theory.) One next computes the spectrum by summing the doppler shifts from all possible pairs of wave vectors. The important point for our purposes is that the bundles of light are presumed to be linear combinations of *plane* waves, each of which has a well-defined wave vector and is therefore of infinite extent normal to the wave vector. Because the plane waves which make up the bundles of light have infinite lateral extent, there can be no finite transit time broadening when using this procedure for estimating the spectral width. The above method was employed by Yeh (*3*) in his discussion of line broadening effects in Brillouin scattering experiments.

Another completely equivalent calculation may be used to compute the broadening. Instead of describing the radiation field by means of a Fourier decomposition into plane waves of different wave vector, $\mathbf{k}$, one describes the radiation field directly in physical space by means of an amplitude distribution function, $P_{(\mathbf{r})}$, which may be complex. This function, and two unique wave vectors $\mathbf{k}_i$ and $\mathbf{k}_s$ specifying the optical axes of the incident and scattered beams respectively, specify the phase and amplitude of the radiation field within the sample volume. This procedure was used by Crosignani et al. to describe the spectrum of light scattered from a diffusely reflecting rotating disk (*4*). Edwards et al. used the same technique in their analysis of the laser doppler velocimeter (*5*). Crosignani (*4*) and Edwards (*5*) each presented data supporting their analyses. When using this technique, it is clear that there is no "uncertainty in wave vector broadening" because the vectors, $\mathbf{k}_i$ and $\mathbf{k}_s$ serve only to define the optic axes and are known precisely.

Several comments on the complex amplitude distribution function $P_{(\mathbf{r})}$ are in order. If one is operating in a region analogous to Fraunhofer diffraction, the wave fronts are planar. (There may be discontinuous changes in phase, however.) In general, the wavefronts are curved and $P_{(\mathbf{r})}$ contains complex, nonlinear phase terms analogous to the situation in Fresnel diffraction. These phase terms must be taken into consideration when computing the broadening in systems operating with very low F numbers or far from the focal plane. In many practical cases they can be ignored. Both Crosignani (*4*) and Edwards (*5*) found agreement between experiment and theory by approximating $P_{(\mathbf{r})}$ with a real function. (See the Appendix for a further discussion of $P_{(\mathbf{r})}$.)

Further insight into the problem can be obtained by recognizing that the two methods of treating the broadening are related through the three-dimensional spatial Fourier decomposition of the radiation field. When one speaks of finite transit time broadening, one is describing the radiation field in physical space. When one speaks of uncertainty in wave vector broadening, one is describing the field in wave vector space. The two are related through the classical laws of physical optics by the three dimensional Fourier transform. The conjugate variables are wave vector and the position vector. From Fourier transform theory there is an uncertainty principle that says

$$\Delta k_x \, \Delta x \approx 1 \qquad (1)$$

That is, constraining the field to a very small region of x, means a large uncertainty in the x component of the wave vector k_x and vice versa. The uncertainties in x and k_x are inextricably related through the laws of Fourier transforms and cannot be treated independently.

The quantum mechanical interpretation of this phenomenon is the well-known Heisenberg uncertainty principle.

Conclusions similar to those reached in this paper have been made using different arguments by Edwards (5), Mayo (6), and Treacy (7).

ACKNOWLEDGMENTS

John C. Angus and Robert V. Edwards acknowledge the support of the National Institute of Arthritis and Metabolic Diseases, National Institutes of Health, Contract PH 43-66-944 and the Office of Naval Research, Contract N00014-67-A-0404-0008.

NOTATION

- a = radius of lens
- F = $f/2a$
- f = focal length of lens
- I = intensity
- $\mathbf{k}$ = wave vector
- P = complex amplitude distribution function
- r = distance from optic axis
- $\mathbf{r}$ = position vector
- u = dimensionless distance from focal plane, Equation (3)
- v = dimensionless distance from optic axis, Equation (4)
- x = Cartesian coordinate
- z = distance from focal plane

Greek Letters

- λ = wave length
- χ = total phase deviation
- ϕ = phase

Subscripts

- i = incident
- o = geometric focus
- s = scattered
- x = Cartesian coordinate

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APPENDIX

A comprehensive description of $P_{(\mathbf{r})}$ must start from the exact solution for the amplitude and phase of a clipped, gaussian beam in the region of the geometric focus. The authors are unaware of such a solution. However, the case of a focused beam from a uniformly illuminated lens has been described in detail by Born and Wolf (8) and Linfoot and Wolf (9) and is illustrative of the principles involved.

From Born and Wolf (8) the phase ϕ of the actual wavefronts is given by

$$\phi_{(u,v)} = \left(\frac{f}{a}\right)^2 u - \chi_{(u,v)} - \pi/2 \tag{2}$$

where

$$u = \left(\frac{2\pi}{\lambda}\right)\left(\frac{a}{f}\right)^2 z \tag{3}$$

$$v = \left(\frac{2\pi}{\lambda}\right)\left(\frac{a}{f}\right) r \tag{4}$$

Born and Wolf's nomenclature is used.

χ may be interpreted as the deviation in phase of the actual wavefronts from ideal plane wave behavior. χ was computed using the methods described by Born and Wolf (8, 9) and tabular values of Lommel functions (10) for various values of u and v. The results are given in Tables 1 and 2. The case of $u = v$ corresponds to the geometric shadow. The radius of the first Airy dark ring is $v = 1.22\pi$.

Along the optic axis in the region of geometric focus, χ changes linearly with u. The total change in χ is -2π for u running from -4π to $+4\pi$. This corresponds to a distance between wave fronts (effective wavelength) that is smaller than the original plane wave wavelength by a factor of $1 - a^2/4f^2$. In typical laser doppler velocimeters $f/2a$ is on the order of 100 which corresponds to a change in effective wavelength of only 6 parts in 10^6.

Off the optic axis the behavior is more complex. In the focal plane there is a discontinuous change of phase of π radians as one crosses each dark ring. At points off the optical axis, but not in the focal plane, the phase fronts become curved. The degree of curvature depends critically upon the focal length and aperture of the lens. The larger the *F*-number ($f/2a$) of the lens, the more closely the wavefronts become planar. At values of $f/2a$ of 100 the wavefronts remain essentially planar at substantial distances from the focal plane. See Tables 1 and 2. Fractional phase deviations, $\chi/(f/a)^2 u$, for other values of $f/2a$ may be easily calculated from Table 2.

Table 1. Phase Deviation and Relative Intensity in Region of Focus of a Uniformly Illuminated Lens

u	v	Total phase deviation, χ, radians	Intensity relative to intensity at geometric focus, I/I_o
0.2	0.2	0.05	~1
2π	0	$\pi/2$	0.404
2π	2π	$3\pi/2$	0.0155

Table 2. Fractional Phase Deviations in the Focal Region of a Uniformly Illuminated Lens of F Number ($f/2a$) Equal to 100

u	v	z, Distance from focal plane, cm.	r, Distance from optic axis, cm.	Fractional Deviation in phase, $\chi/(f/a)^2 u$
0.2	0.2	8×10^{-2}	4×10^{-4}	6.4×10^{-6}
2π	0	2.51	0	6.4×10^{-6}
2π	2π	2.51	1.26×10^{-2}	1.93×10^{-5}